Efficient Eigenvalue Assignment for Large Space Structures

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A novel and efficient approach for the eigenvalue assignment of large, first-order, time-invariant systems is developed using full-state feedback and output feedback. The full-state feedback approach basically consists of three steps. First, a Schur decomposition is applied to triangularize the state matrix. Second, a series of coordinate rotations (Givens rotations) is used to move the eigenvalue to be reassigned to the end of the diagonal of its Schur form. Third, the eigenvalue is moved to the desired location by a full-state feedback without affecting the remaining eigenvalues. The second and third steps can be repeated until all the assignable eigenvalues are moved to the desired locations. Given the freedom of multiple inputs, the feedback gain matrix is calculated to minimize an objective function composed of a gain matrix norm and/or a robustness index of the closed-loop system. An output feedback approach is also developed using similar procedures as for the full-state feedback wherein the maximum allowable number of eigenvalues may be assigned. Numerical examples are given to demonstrate the feasibility of the proposed approach.

Introduction

TRADITIONALLY, eigenvalue and eigenvector assignment methods have been used as control design tools for linear time-invariant systems. To date, numerous procedures and algorithms covering the various aspects of the problem have been outlined in the literature. Wonham[1] established the equivalency of the controllability of linear multi-input systems and the assignability of their eigenvalues via full-state feedback. Moore[2] and Klien and Moore[3] identified the flexibility beyond the eigenvalue assignment in multi-input systems by characterizing the attainable closed-loop eigenvector space. Davison,[4,5] Kimura,[6] Srinathkumar,[7] and other researchers extended the idea of eigenvalue assignment to systems with partial state feedback and established the limitations imposed by such feedbacks. Within the past decade, several iterative and noniterative algorithms have been derived to exploit the freedom offered beyond eigenvalue assignment by multi-inputs and multi-outputs, in order to either improve the performance of the closed-loop system or to minimize the required control effort. Among the class of iterative methods is Kautsky's algorithm,[8] which iteratively minimizes some robustness measure of the closed-loop system in terms of the conditioning of the closed-loop modal matrix through an orthogonal projection approach. Direct nonlinear programming techniques were used to minimize scalar robustness measures such as closed-loop conditioning[9,10] and closed-loop normality indices.[11] In the class of noniterative methods, a sequential algorithm by Varga[12] based on real Schur and QR decompositions for pole assignment with full state feedback can be identified. A recent algorithm by Juang et al.[13] and Rew et al.[14] can also be named wherein closed-loop eigenvectors are chosen to maximize their orthogonal projection to the open-loop eigenvector matrix or its closest unitary matrix in order to maximize the robustness of the closed-loop system. Many of the existing methods can-

not maintain computational efficiency when a small number of eigenvalues are assigned compared to the order of the system. Recent trends toward the erection and deployment of large flexible structures having thousands of degree of freedoms require new techniqes that are computationally stable and more efficient.

In this paper, a novel approach for the eigenvalue assignment of large, first-order, time-invariant systems is developed. The approach is particularly efficient when the number of assigned eigenvalues is smaller than the order of the system. The system is assumed to be fully controllable and observable. If full-state feedback is available, the approach starts with a Schur decomposition to triangularize the state matrix followed by a series of coordinate rotations (Givens rotations) to arbitrarily place the eigenvalues of the state matrix on the diagonal of its Schur form. With the eigenvalue or pair of complex conjugate eigenvalues to be assigned positioned at the end of the diagonal, a feedback gain matrix in the transformed coordinates is chosen to place the specified eigenvalue(s) without changing the remaining eigenvalues on the diagonal. Moreover, given the freedom of multi-input systems, the gain matrix is also computed to minimize an objective function composed of a gain matrix norm and/or a robustness index of the closed-loop system. For the output feedback case, there are two steps involved in the eigenvalue assignment. First, some of the eigenvalues are assigned through conventional output feedback algorithms.[4,5,7,13,14] Second, an output gain matrix is derived, again with the aid of Schur decomposition as well as coordinate transformations, to place the remaining eigenvalues while keeping the previously assigned ones unchanged. The output feedback algorithm can assign the maximum allowable number of eigenvalues. Because of the orthogonal nature of the transformations, both full-state and output feedback algorithms are computationally stable. Numerical examples are given to demonstrate the feasibility of both algorithms.

Full-State Feedback

The dynamics of a linear flexible space structure can be represented by a first-order, linear, time-invariant system as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$

$$y(t) = Cx(t) \qquad (2)$$

where A is an $n \times n$ state matrix, B is an $n \times m$ control input influence matrix, C is a $p \times n$ output influence matrix, $x(t)$ is

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an $n \times 1$ state vector, $y(t)$ is a $p \times 1$ output measurement vector, and $u(t)$ is an $m \times 1$ vector of the control inputs. Assume that the system represented by Eq. (1) is fully controllable and a full-state feedback is used to design a constant feedback controller with gain matrix G of dimension $m \times n$, such that $u(t) = Gx(t)$. The gain matrix G is to be chosen in such a way that, for example, a pair of eigenvalues of the open-loop state matrix A are assigned to desired values without shifting the remaining eigenvalues.

The eigenvalue assignment method for full-state feedback starts with applying the Schur transformation[15] to Eq. (1), i.e.,

$$x = Vx_1 \tag{3}$$

and premultiplying the resulting equation by V^H to yield

$$\dot{x}_1 = V^H A V x_1 + V^H B G V x_1 \tag{4}$$

where V is an $n \times n$ unitary Schur matrix and $(\ )^H$ denotes complex conjugate transpose. The state matrix in the transformed coordinates, namely, $V^H A V$ is an upper triangular matrix with the eigenvalues of A on its diagonal, i.e.,

$$V^H A V = \begin{bmatrix} \lambda_1 & X & \cdots & X \\ 0 & \lambda_2 & \cdots & X \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Assume that the pair of eigenvalues to be assigned occupy the ith and jth ($i < j$) positions on the diagonal. Observe from Eq. (4) that it is possible to derive a gain matrix GV (in the transformed coordinates) such that the feedback portion of the closed-loop state matrix, namely, $V^H B G V$ is an upper triangular matrix with all its diagonal elements zero except the last. Hence, with such a gain matrix, the last eigenvalue of matrix A (the last element on the diagonal of $V^H A V$) can be assigned to a desired value without shifting the remaining eigenvalues of A. In order to extend this technique to assign any arbitrary eigenvalue λ_i (the ith on the diagonal of $V^H A V$), for example, a series of appropriate coordinate transformations (coordinate system rotations) must be used to move λ_i to the last position on the diagonal as follows. There are basically two steps involved.

1) Apply an appropriate unitary coordinate transformation R_i such that $x_1 = R_i x_{R_i}$ to switch the position of λ_i and λ_{i+1} on the diagonal. Such a transformation may be defined by

$$R_i = \begin{bmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ 0 & 1 & 0 & & & & \vdots \\ \vdots & & \ddots & & & & \\ & & & c_i & s_i^* & & \\ & & & -s_i & c_i & & \\ 0 & \cdots & & & & \ddots & 1 \end{bmatrix} \tag{5}$$

The matrix R_i is essentially a unitary Givens rotation.[15] The c_i and s_i are, respectively, real and complex scalar parameters defined such that

$$c_i^2 + s_i s_i^* = 1 \tag{6}$$

and the superscript * denotes complex conjugate. The required parameters c_i and s_i of Eq. (5) may be determined from

$$c_i^2 = \frac{1}{1 + \left\{ \left| (\lambda_i - \lambda_{i+1})(\lambda_i - \lambda_{i+1})^* \right| / bb^* \right\}} \tag{7}$$

$$s_i = \frac{(\lambda_i - \lambda_{i+1}) c_i}{b} \tag{8}$$

in which b is the $(i, i+1)$ element of the matrix $V^H A V$. Note that if $b = 0$, then c_i and s_i are, respectively, set to 0 and 1 in Eq. (5). The state matrix in the transformed coordinates becomes

$$R_i^H V^H A V R_i = \begin{bmatrix} \lambda_1 & \cdots & & & \cdots & X \\ \vdots & \ddots & & & & \vdots \\ & & \lambda_{i+1} & & & \\ & & & \lambda_i & & \\ \vdots & & & & \ddots & \\ 0 & \cdots & & & & \lambda_n \end{bmatrix} \tag{9}$$

The unitary transformation R_i of Eq. (5) becomes identity when λ_i and λ_{i+1} are not distinct. In such a case, λ_{i+1} can be assigned instead of λ_i, and therefore, there is no need for applying the transformation R_i.

2) Repeat the procedures of the first step $n - i$ times in order to move λ_i to the last position on the diagonal. The dynamics of the system in the final transformed coordinates may be written as

$$\dot{x}_2 = L^H A L x_2 + L^H B G L x_2 \tag{10}$$

or

$$\dot{x}_2 = \bar{A} x_2 + \bar{B} \bar{G} x_2 \tag{11}$$

where L is the composite unitary transformation that moves λ_i to the end of the diagonal and is given as follows:

$$L = V R_i R_{i+1} \cdots R_{n-1} \tag{12}$$

and

$$\bar{A} = L^H A L; \qquad \bar{B} = L^H B; \qquad \bar{G} = GL \tag{13}$$

Because of the particular structure of R_i, the pre- and post-multiplication of the matrices A and B by R_i^H and R_i do not require full multiplication of these matrices since only two columns or two rows are affected at a time. Consequently, the multiplications are carried out for the appropriate rows and columns only. The unitary nature of the transformations is particularly attractive since it ensures stable numerical computations even for large-order systems. Note that as λ_i is moved to the end of the diagonal, all the eigenvalues below λ_i move one position toward the top of the diagonal, i.e., the jth eigenvalue λ_j occupies the $(j-1)$th position.

To ensure the assignment of the desired value for λ_i, without affecting the remaining eigenvalues, the matrix $\bar{G}$ should be chosen in such a way that $\bar{B}\bar{G}$ is an upper triangular matrix with its diagonal zero except for its end element. Such a gain matrix may be of the form

$$\bar{G} = (0 \mid \bar{g}_1) \tag{14}$$

where $\bar{g}_1$ is an $m \times 1$ vector. Obviously, with such a choice, $\bar{B}\bar{G}$ would be a null matrix except for its last column. The local gain matrix G is related to the vector $\bar{g}_1$ from Eq. (13)

$$G = \bar{g}_1 L_n^H \tag{15}$$

where L_n represents the last column of the transformation matrix L. Equation (15) may also be written in a vector form

$$\hat{g}_1 = F \bar{g}_1 \tag{16}$$

in which $\hat{g}_1$ is an $(n \times m) \times 1$ vector representing the packed columns of the gain matrix G, and F is an $(n \times m) \times m$ matrix defined as

$$F = \begin{bmatrix} L_{1,n}^* \hat{I} \\ L_{2,n}^* \hat{I} \\ \vdots \\ L_{n,n}^* \hat{I} \end{bmatrix} \qquad (17)$$

here, $L_{i,j}$ represents the (i,j) element of the matrix L, and $\hat{I}$ is an $m \times m$ identity matrix.

Assume that μ_i is the desired closed-loop value for λ_i, then $\bar{g}_1$ should be determined such that

$$b_n \bar{g}_1 = \mu_i - \lambda_i \qquad (18)$$

in which b_n represents the last row of the current matrix $\bar{B}$. Any given solution of $\bar{g}_1$ that satisfies Eq. (18) will then produce a local gain matrix G from Eq. (15) that will move the eigenvalue λ_i to the desired eigenvalue μ_i. Now, depending on the number of control inputs, an optimum solution for $\bar{g}_1$ corresponding to a minimum gain design may be formulated. The complex matrices and vectors in Eq. (16) can be expanded in terms of their respective real and imaginary components, to yield

$$F_R \bar{g}_{1_R} - F_I \bar{g}_{1_I} = \hat{g}_{1_R} \qquad (19)$$

$$F_I \bar{g}_{1_R} - F_R \bar{g}_{1_I} = \hat{g}_{1_I} \qquad (20)$$

where the subscripts R and I denote the real and imaginary components, respectively.

Vector $\bar{g}_1$ is now chosen to minimize a cost function J_1

$$J_1 = (\hat{g}_{1_R} + \hat{g}_0)^T (\hat{g}_{1_R} + \hat{g}_0) \qquad (21)$$

subjected to the constraint of Eq. (18) that assigns the eigenvalue λ_i. The $\hat{g}_0$ is a real $(n \times m) \times 1$ vector representing the packed columns of the current global gain matrix (prior to the assignment of λ_i). The current global gain matrix $\hat{g}_0$ is obtained by adding all the local gain matrices that move all the eigenvalues λ_j, $j = 1, \ldots, i-1$ to the desired eigenvalues μ_j, $j = 1, \ldots, i-1$, and keep the remaining eigenvalues unchanged. Note that $\hat{g}_0$ is a null vector if λ_i is the first eigenvalue to be assigned. A transpose is denoted by $(\ )^T$.

Using Eq. (19) in Eq. (21), the cost function J_1 may be redefined in terms of the vectors $\bar{g}_{1_R}$ and $\bar{g}_{1_I}$ as follows

$$J_1 = (F_R \bar{g}_{1_R} - F_I \bar{g}_{1_I} + \hat{g}_0)^T (F_R \bar{g}_{1_R} - F_I \bar{g}_{1_I} + \hat{g}_0)$$

or

$$J_1 = z_1^T Q_1 z_1 + 2Q_2 z_1 + \hat{g}_0^T \hat{g}_0 \qquad (22)$$

where $z_1 = (\bar{g}_{1_R}^T, \bar{g}_{1_I}^T)^T$, and

$$Q_1 = \begin{pmatrix} F_R^T F_R & -F_R^T F_I \\ -F_I^T F_R & F_I^T F_I \end{pmatrix}; \qquad Q_2 = \hat{g}_0^T (F_R^T \quad -F_I^T)$$

The constraint of Eq. (18) can similarly be expressed in terms of the augmented vector z_1, i.e.,

$$E z_1 = f_1 \qquad (23)$$

in which

$$E = \begin{pmatrix} b_{n_R} & -b_{n_I} \\ b_{n_I} & b_{n_R} \end{pmatrix}; \qquad f_1 = \begin{pmatrix} \mathrm{Re}(\mu_i - \lambda_i) \\ \mathrm{Im}(\mu_i - \lambda_i) \end{pmatrix}$$

The cost function given in Eq. (22) along with the constraint of Eq. (23) represents an optimization problem. The optimal solution yields a gain matrix that would assign the ith eigenvalue λ_i to a desired value μ_i, keep the other eigenvalues of the current state matrix unchanged, and minimize the Frobenius norm of the global gain matrix. The choice of the cost function as to minimize the norm of the gain matrix is particularly practical for space application where available power is so limited. However, other cost indices minimizing such quantities as conditioning and robustness measures could also be implemented with the proposed eigenvalue assignment technique.

Using the Lagrangian multipliers, the optimization problem, Eqs. (22) and (23), can be reduced to the solution of a system of $(2m + 2)$ simultaneous equations given by

$$\begin{pmatrix} Q_1 & E^T \\ E & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -Q_2 \\ f \end{pmatrix} \qquad (24)$$

where γ_1 is the 2×1 vector of the Lagrangian multipliers. The solution of Eq. (24) becomes

$$z_1 = -Q_1^\dagger \big[Q_2 - E^T (E Q_1^\dagger E^T)^\dagger \{ f + E Q_1^\dagger Q_2 \} \big] \qquad (25)$$

where $(\ )^\dagger$ denotes a pseudoinverse. Both the global gain matrix $\hat{g}_0$ and the state matrix $\bar{A}$ are updated after assigning λ_i, i.e.,

$$\hat{g}_0 \equiv \hat{g}_0 + \hat{g}_1 \qquad (26)$$

$$\bar{A} \equiv \bar{A} + \bar{B}\bar{G} \qquad (27)$$

Obviously, any practical solution for the gain matrix $\hat{g}_1$ must be real, but the solution of the optimization problem is, in general, complex. This problem is overcome by adding an additional constraint to the optimization problem to ensure that the global gain matrix is real when the complex conjugate of the eigenvalue λ_i is being assigned. Such a constraint may be expressed as

$$F_I \bar{g}_{2_R} + F_R \bar{g}_{2_I} = -\hat{g}_{1_I} \qquad (28)$$

where $\bar{g}_2$ represents the column of the gain matrix (in the transformed coordinates) associated with the second eigenvalue (the complex conjugate of the eigenvalue λ_i) assignment, see Eq. (14), and $\hat{g}_{1_I}$ is the imaginary part of the gain matrix obtained from the Eq. (25) in the assignment of λ_i.

Now, to assign the eigenvalue λ_i^* to its desired value μ_i^*, a series of coordinate transformations given by Eq. (5) are employed to move the eigenvalue to the end of the diagonal. Then the required gain matrix is determined from the solution of the following optimization problem:

Minimize a cost function J_2

$$J_2 = z_2^T Q_1 z_2 + 2Q_2 z_2 + \hat{g}_0^T \hat{g}_0 \qquad (29)$$

subject to the constraint of Eq. (28) and

$$E z_2 = f_2 \qquad (30)$$

where $z_2 = (\bar{g}_{2_R}^T, \bar{g}_{2_I}^T)^T$, the matrices Q_1, Q_2, and E have been defined previously, and

$$f_2 = \begin{pmatrix} \mathrm{Re}(\mu_i^* - \lambda_i^*) \\ \mathrm{Im}(\mu_i^* - \lambda_i^*) \end{pmatrix}$$

The optimization problem of Eqs. (28–30) can easily be solved using the Lagrangian multipliers approach, as discussed previously. However, in many cases the constraints of Eqs. (28) and (30) do not allow any freedom for optimization. In such cases, the solution of the gain matrix is given from Eq. (28) as

$$z_2 = -(F_I \quad F_R)^\dagger \hat{g}_{1_I} \qquad (31)$$

or

$$\begin{bmatrix} \bar{g}_{2_R} & \bar{g}_{2_I} \end{bmatrix} = -G_I \begin{pmatrix} \mathrm{Re}\{L_n^H\} \\ \mathrm{Im}\{L_n^H\} \end{pmatrix}^\dagger \tag{32}$$

where G_I is an $m \times n$ matrix denoting the unpacked form of the vector $\hat{g}_{1_I}$. Obviously, the latter form of the solution is computationally more efficient. It can be shown that Eqs. (31) and (32) are exactly satisfied when each complex conjugate pair of open-loop eigenvalues are assigned to complex conjugate or real values.

The eigenvalue assignment technique discussed so far assigns one eigenvalue at a time while attempting to minimize the current norm of the gain matrix. However, an alternative procedure may also be considered wherein one pair of complex conjugate eigenvalues are assigned at each step. To accomplish this the pair of eigenvalues are moved to the end of the diagonal via unitary transformations. Then, the gain matrix $\bar{G}$ (in the transformed coordinates) is chosen as

$$\bar{G} = \begin{pmatrix} 0 & | & \bar{g} \end{pmatrix} \tag{33}$$

here, $\bar{g}$ is an $m \times 2$ matrix to be computed from an optimization problem. A cost function similar to Eq. (22) can be used subject to a constraint similar to Eq. (28) to ensure a real solution for the gain matrix G. The pair of eigenvalues can be placed to desired values by

$$b_{n-1}\bar{g}_1 + b_n\bar{g}_2 = 2\mathrm{Re}(\lambda_i - \mu_i) \tag{34}$$

$$[b_{n-1}\bar{g}_1][b_n\bar{g}_2] - [b_{n-1}\bar{g}_2][b_n\bar{g}_1] + \lambda_i b_{n-1}\bar{g}_1 + \lambda_i^* b_n\bar{g}_2$$
$$= b + |\mu_i|^2 - |\lambda_i|^2 \tag{35}$$

in which b_{n-1} and b_n represent the $(n-1)$th and nth rows of $\bar{B}$, respectively, g_1 and g_2 denote the columns of $\bar{g}$, and b has been defined previously. Because of the quadratic nature of Eq. (35), the minimization problem cannot be solved analytically, but its solution may be obtained with relative ease through classical quadratic programming routines.

The procedures developed above may be repeated to assign some or all the eigenvalues of the state matrix. The sequential aspect of the algorithm is most attractive for eigenvalue assignment of large-order systems in which only a few numbers of poles are required to be assigned. It should be noted that the state matrix $\bar{A}$, the control influence matrix $\bar{B}$, and the composite transformation matrix L need to be updated after each unitary coordinate transformation. Furthermore, the state matrix $\bar{A}$, as well as the global gain matrix $\hat{g}_0$, must be updated after each individual eigenvalue assignment according to Eqs. (26) and (27).

Again it is stressed that the eigenvalue-assignment technique discussed can assign any number of the eigenvalues of the closed-loop state matrix while minimizing the norm of the gain matrix required in this effort at each step. However, the procedure neglects to consider the robustness of the closed-loop system, defined as the sensitivity of the resulting closed-loop system to parametric perturbations. These perturbations are generally attributed to manufacturing errors, modeling uncertainty, or environmental changes. Although the minimum-gain design that correpsonds, in a general sense, to a minimum-control-energy design is desirable for space applications (due to limited available power), some consideration should be given to the robustness of the closed-loop system, particularly in large-order complex space structures where such perturbations may be significant.

The robustness of a system is commonly measured in terms of the conditioning of the state matrix, or specifically, in terms of the condition number $c(\Psi)$ of the modal matrix Ψ bearing the eigenvectors of the state matrix. The matrix Ψ satisfies the

eigenvalue problem

$$A\Psi = \Psi\Delta \tag{36}$$

in which, when the eigenvectors are linearly independent, Δ is the diagonal matrix of the eigenvalues of A. The condition number $c(\Psi)$ is defined as

$$c(\Psi) = |\Psi|_2 |\Psi^{-1}|_2 \tag{37}$$

where $|\ \ |_2$ indicates a 2 norm.

It is observed from Eq. (14) that during the assignment of an eigenvalue, only the last column of the state matrix is affected by the feedback return vector. More specifically, since the last element of the feedback return vector that assigns the desired eigenvalue does not affect the eigenvectors of the state matrix, thus, only the first $(n-1)$ elements of vector $\bar{B}\bar{g}_1$ affect the eigenvectors. This suggests that the deviation from open-loop conditioning can be minimized indirectly through minimizing the norm of the vector corresponding to the first $(n-1)$ elements of the feedback return $\bar{B}\bar{g}_1$. Assuming that the condition number $c(\Psi)$ is good to begin with, the cost functions J_1 [Eq. (22)] and J_2 [Eq. (29)] can be redefined to accommodate some consideration for conditioning or robustness of the system. Define the new cost functions $\bar{J}_1$ and $\bar{J}_2$ as

$$\bar{J}_1 = \alpha J_1 + \beta \left[\bar{g}_1^H \hat{B}^H \hat{B} \bar{g}_1 \right] \tag{38}$$

$$\bar{J}_2 = \alpha J_2 + \beta \left[\bar{g}_2^H \hat{B}^H \hat{B} \bar{g}_2 \right] \tag{39}$$

where $\hat{B}$ is an $(n-1) \times m$ matrix representing the first $(n-1)$ rows of the matrix $\bar{B}$; α and β are non-negative arbitrary weighting constants. The newly defined cost functions $\bar{J}_1$ and $\bar{J}_2$ are still quadratic and subject to the same constraints as J_1 and J_2, respectively. Furthermore, the optimization problems posed by these cost functions can be solved using the Lagrangian multipliers approach as discussed previously. Observe that if β is chosen to be zero, the optimization problem of Eqs. (38) and (39) reduces to the minimium-gain design of Eqs. (22) and (29). On the other hand, if α is chosen to be zero, the optimization involves a minimization of Frobenius norms of the vectors $\hat{B}\bar{g}_1$ and $\hat{B}\bar{g}_2$, thereby, indirectly keeping the conditioning of the resulting closed-loop system as close to the open-loop conditioning as possible. Obviously, any optimum choice for α and β, which results in a robust closed-loop system that is obtained through moderate control effort, is generally problem dependent and may only be obtained using heuristic search procedures.

Output Feedback

Having in mind that full-state feedback may not be available for implementation in many real applications, an eigenvalue assignment procedure using output feedback is developed. Davison[4,5] has shown that for (almost all) fully controllable and observable systems $\min(m+p-1,n)$ eigenvalues of the system may be arbitrarily assigned with real gains. Kimura[6] has established that for (almost all) fully controllable and observable systems with m inputs and p outputs all the eigenvalues may be arbitrarily assigned provided that $m+p-1 \geq n$. The dynamics of the system with output feedback is given as

$$\dot{x}(t) = [A + BGC]x(t) \tag{40}$$

where G is an $m \times p$ output feedback gain matrix, and A, B, C, and $x(t)$ have been defined previously. Any of the output feedback algorithms outlined in the literature[4,5,7,13,14] may be used to assign $\bar{m}$ pairs of eigenvalues, where $2\bar{m} \leq m-1$. A good choice might be the numerically sound algorithm outlined in Ref. 13, which attempts to maximize a closed-loop robustness measure in addition to the eigenvalue assignment.

Without going into the details of this procedure, let G_1 denote the appropriate gain matrix. Then, Eq. (40) may be written as

$$\dot{x}(t) = [\bar{A} + BG_2C]x(t) \tag{41}$$

in which $\bar{A} = A + BG_1C$ and G_2 denotes the portion of the gain matrix that places the remaining assignable eigenvalues to their desired values while keeping those assigned by G_1 unchanged. Performing the Schur transformation of the state matrix $\bar{A}$ and rearranging the previously assigned eigenvalues to the end of the diagonal via unitary Givens transformations yields

$$\dot{\hat{x}}(t) = [\hat{A} + \hat{B}G_2\hat{C}]\hat{x}(t) \tag{42}$$

where $x(t) = L\hat{x}(t)$, $\hat{A} = L^T\bar{A}L$, $\hat{B} = L^TB$, $\hat{C} = CL$, and L is the cumulative transformation matrix defined as $L = UR_1 \cdots R_s$. Now, $\hat{A}$ is an upper triangular matrix with eigenvalues of $\bar{A}$ on its diagonal and the $\bar{m}$ pairs of eigenvalues assigned via G_1 at the end of the diagonal. One of the major shortcomings of eigenvalue assignment algorithms via output feedback is that they cannot guarantee the stability of the resulting closed-loop system. It might be more practical to assign a lesser number of eigenvalues than the maximum number $\min(m + p - 1, n)$ in order to preserve the stability of the resulting closed-loop system. Therefore, one might consider keeping up to $\bar{m}$ pairs of the most sensitive open-loop eigenvalues unchanged instead of assigning them new values [see Eq. (41)].

Let $\hat{B}_{\bar{m}}$ denote the last $2\bar{m}$ rows of $\hat{B}$. Then, it is obvious that if the columns of G_2 are chosen to lie in the right null space of $\hat{B}_{\bar{m}}$, the feedback return matrix $\hat{B}G_2\hat{C}$ will not affect the $\bar{m}$ pairs of eigenvalues at the end of the diagonal of $\hat{A}$. Defining a set of orthonormal basis spanning the right null space of matrix $\hat{B}_{\bar{m}}$ by Ψ, i.e.,

$$\hat{B}_{\bar{m}}\Psi = 0 \tag{43}$$

the gain matrix G_2 can be expanded in terms of the null basis Ψ

$$G_2 = \Psi q \tag{44}$$

in which q is an $r \times p$ coefficient matrix, and r denotes the dimension of the null basis. The eigenvalue assignment problem is now reduced to finding a set of coefficients q that assigns the remaining p eigenvalues. This is equivalent to assigning the p eigenvalues of the following subsystem

$$\dot{\hat{x}}_r(t) = [\hat{A}_r + \hat{B}_r\Psi q\hat{C}_r]\hat{x}_r \tag{45}$$

where $\hat{A}_r$, $\hat{B}_r$, and $\hat{C}_r$ are submatrices composed of the first $(n - 2\bar{m})$ rows and columns of $\hat{A}$, the first $(n - 2\bar{m})$ rows of $\hat{B}$, and the first $(n - 2\bar{m})$ columns of $\hat{C}$, respectively. Here, $\hat{x}_r$ denotes the first $(n - 2\bar{m})$ elements of $\hat{x}$. Since all the variables in Eq. (45) are complex, the task of assigning the desired eigenvalues may be quite cumbersome. However, due to the unique nature of the Schur vectors a more amenable and computationally efficient companion system may be considered instead as follows:

$$\dot{z}(t) = [A_c + B\Psi qC]z(t) \tag{46}$$

where A_c is an $n \times n$ defined as

$$A_c = L\begin{pmatrix} \hat{A}_r & 0 \\ 0 & 0 \end{pmatrix}L^T$$

and $z(t)$ is an $n \times 1$ companion state vector. The eigenvalues of A_c are the same as eigenvalues of $\hat{A}_r$ except for $\bar{m}$ pairs of additional zero eigenvalues, which will similarly not be affected by the feedback return $B\Psi qC$. All the variables in Eq. (46) are real except Ψ and q. Since only real-valued gain ma-

trices are meaningful, the gain matrix in Eq. (46) is replaced by its real part, such that

$$\dot{z}(t) = [A_c + B[\Psi_R q_R - \Psi_I q_I]C]z(t) \tag{47}$$

Now every variable in Eq. (47) is real. The eigenvalue problem for Eq. (47) is given

$$[A_c + B[\Psi_R q_R - \Psi_I q_I]C]\eta_k = \eta_k\bar{\mu}_k \tag{48}$$

where η_k and $\bar{\mu}_k$, respectively, denote the kth eigenvector and desired eigenvalue. Expanding Eq. (48) in terms of real and imaginary parts of η_k and $\bar{\mu}_k$ and rearranging the results in matrix form yields

$$\Gamma_k\begin{bmatrix} \eta_{kR} \\ \eta_{kI} \\ q_RC\eta_{kR} \\ q_IC\eta_{kR} \\ q_RC\eta_{kI} \\ q_IC\eta_{kI} \end{bmatrix} \equiv \Gamma_k\phi_k = 0 \tag{49}$$

where

$$\Gamma_k = \begin{pmatrix} A_c - \mu_{kR}I_n & \mu_{kI}I_n & B\hat{\Psi} & 0 \\ -\mu_{kI}I_n & A_c - \mu_{kR}I_n & 0 & B\hat{\Psi} \end{pmatrix}$$

and $\hat{\Psi} = [\Psi_R \quad \Psi_I]$. Equation (49) is satisfied for each pair of complex conjugate eigenvalues. Let ν_k denote an orthonormal basis for the solution of Eq. (49) that obviously spans the null space of Γ_k. Such a solution may be obtained via the singular value decomposition or QR decomposition of Γ_k.[13-15] If s pairs of eigenvalues are to be assigned, then the solution of the homogeneous equation of Eq. (49) may be written for the kth pair $(k = 1, \ldots, s)$ as

$$\Gamma_k\phi_k \equiv \Gamma_k\begin{bmatrix} \phi_{kR} \\ \phi_{kI} \\ \phi_{kRR} \\ \phi_{kRI} \\ \phi_{kIR} \\ \phi_{kII} \end{bmatrix} = \Gamma_k\begin{bmatrix} \nu_{kR} \\ \nu_{kI} \\ \nu_{kRR} \\ \nu_{kRI} \\ \nu_{kIR} \\ \nu_{kII} \end{bmatrix}c_k = 0 \tag{50}$$

where the null basis ν_k has been partitioned into six components according to Eq. (49), and c_k are the appropriate coefficients for the basis ν_k. Comparison of Eqs. (49) and (50) yields that the matrices q_R and q_I must satisfy

$$q_RC[\phi_{kR}, \phi_{kI}] = [\phi_{kRR}, \phi_{kIR}]; \qquad k = 1, \ldots, s \tag{51}$$

$$q_IC[\phi_{kR}, \phi_{kI}] = [\phi_{kRI}, \phi_{kII}]; \qquad k = 1, \ldots, s \tag{52}$$

It is observed from Eq. (50) that any freedom provided by the multi-inputs and multi-outputs is imbedded in the coefficients c_k. This freedom can be exploited to achieve better closed-loop performance criterion such as robustness in terms of conditioning of the modal matrix[13] or the normality measures,[11] or minimizing a weighted norm of the gain matrix. However, as long as the coefficients c_k are chosen to generate a set of linearly independent eigenvectors, a solution for the gain matrix can be obtained. Equations (51) and (52) can be written in

a general matrix form

$$q_R \Phi = \hat{\Phi} \qquad (53)$$

$$q_I \Phi = \bar{\Phi} \qquad (54)$$

in which

$$\Phi = C[\phi_{1R}, \phi_{1I}, \ldots, \phi_{sR}, \phi_{sI}]$$

$$\hat{\Phi} = [\phi_{1RR}, \phi_{1IR}, \ldots, \phi_{sRR}, \phi_{sIR}]$$

$$\bar{\Phi} = [\phi_{1RI}, \phi_{1II}, \ldots, \phi_{sRI}, \phi_{sII}]$$

The solution for q_R and q_I is then given as

$$q_R = \hat{\Phi}[\Phi]^\dagger \qquad (55)$$

$$q_I = \bar{\Phi}[\Phi]^\dagger \qquad (56)$$

Note that the solutions given in Eqs. (55) and (56) are unique or minimum norm depending on the number of pairs assigned. With q_R and q_I determined, the gain matrix G_2 is computed as

$$G_2 = \Psi_R q_R - \Psi_I q_I \qquad (57)$$

and the actual gain matrix G is obtained as

$$G = G_1 + G_2 \qquad (58)$$

It can be shown that the columns of the gain matrix G_2 satisfy the orthogonality condition of Eq. (43) provided that $\bar{m}$ pairs of eigenvalues assigned previously are either real or complex conjugates. The procedure developed here can assign up to $\min(m + p - 1, n)$ eigenvalues with partial state feedback. A different but similar procedure can also be used wherein first, $\bar{p}(2\bar{p} \le p - 1)$ pairs of eigenvalues assigned via output feedback algorithms are relocated to the top of the diagonal of the state matrix $\bar{A}$ of Eq. (41). Then, the rows of the gain matrix G_2 are designed to be orthogonal to the columns of matrix $\hat{C}_{\bar{p}}$ that is formed from the first $2\bar{p}$ columns of $\hat{C}$. Without going into further details, a companion system similar to Eq. (46) may be used

$$\dot{z}(t) = [A_c + Bh\Upsilon C]z(t) \qquad (59)$$

where Υ represents a set of orthonormal basis spanning the left null space of matrix $\hat{C}_{\bar{p}}$, and h is the corresponding coefficient matrix. Since the gain matrix G_2 must be real, Eq. (59) is rewritten in a real form

$$\dot{z}(t) = \left[A_c + B[h_R \Upsilon_R - h_I \Upsilon_I]C\right]z(t) \qquad (60)$$

The eigenvalue assignment problem is reduced to finding the matrices h_R and h_I to assign the remaining assignable eigenvalues. This can be achieved following the approach given in Eqs. (48-56) except that a left eigenvalue problem should be considered instead of a right one [see Eq. (48)].
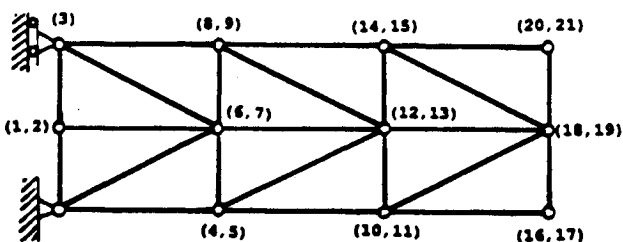

Fig. 1 Flexible truss structure.

Numerical Example

The eigenvalue assignment techniques for full-state feedback and output feedback are demonstrated through their application to a finite element model of a truss structure shown in Fig. 1. The model consists of 21 degrees of freedom (see Fig. 1) with no actuator dynamics considered. The actuation is provided by six actuators acting along the degrees of freedom 3, 7, 10, 11, 20, and 21. Proportional structural damping is assumed. The second-order finite element model is converted into a first-order system with 42 state variables. The state matrix and the control influence matrix are expressed by

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}; \qquad B = \begin{pmatrix} 0 \\ M^{-1}H \end{pmatrix}$$

where M, D, and K are, respectively, the mass, damping, and stiffness matrices, I is an $n \times n$ identity matrix, and H is the control influence matrix in the second-order system.

Full-State Feedback

The first five modes of the truss structure (five pairs of complex conjugate eigenvalues in the state-space model) are assigned to desired locations through full-state feedback. The open-loop eigenvalues and the corresponding closed-loop eigenvalues are summarized in Table 1. The real parts of the closed-loop eigenvalues are chosen to be smaller than -0.5 to satisfy certain pointing requirements, whereas the imaginary parts are basically chosen to be close to the corresponding open-loop values. The procedure for the assignment of the eigenvalues is summarized as follows.

1) After applying the Schur transformation, all the eigenvalues to be assigned are moved to the end of the diagonal via the unitary Givens rotations of Eq. (5).

2) Observe from Eq. (14) that the assignability of the eigenvalues at the end of the diagonal is dictated by the last row of the control influence matrix b_n. In fact, the norm of b_n might be used as a measure of assignability of the corresponding eigenvalue. Generally, some eigenvalues are more assignable than others. It is reasonable to assign the more assignable eigenvalues first since after each eigenvalue assignment, the assignability of the remaining eigenvalues changes. Accordingly, a search for the most assignable eigenvalue is implemented by comparing the value of the norm of b_n for each eigenvalue remaining to be assigned. The eigenvalue with the highest degree of assignability is then assigned along with its complex conjugate pair.

Steps 1 and 2 are repeated until all the desired eigenvalues are assigned.

Two different gain matrix designs are investigated. The first design is associated with the minimum-norm design of Eq. (22), and the second design corresponds to the optimized design of Eq. (29). The results of the eigenvalue assignment using the minimum-norm design are

Norm of the gain matrix $|G|_F$: 29.1
Open-loop condition number $c(\Psi)$: 37.42
Closed-loop condition number $c(\Psi)$: 53.25

The norm of the gain matrix is quite reasonable, whereas the closed-loop condition number is relatively higher than the open-loop indicating that robustness could become an issue with this design. The same eigenvalues are also assigned using the second design, which attempts to maintain the conditioning of the closed-loop system close to the open-loop, yielding

Norm of the gain matrix $|G|_F$: 30.1
Open-loop condition number $c(\Psi)$: 37.42
Closed-loop condition number $c(\Psi)$: 37.90

The norm of the gain matrix is slightly higher than that of the previous case; however, the closed-loop conditioning is much closer to the open-loop one indicating that the second design may be more feasible. Generally speaking, the mini-

Table 1   Open- and Closed-Loop Eigenvalues

Open-loop eigenvalues	Closed-loop eigenvalues
$-0.0064 \pm 1.2617i$	$-0.5000 \pm 2.0000i$
$-0.0556 \pm 3.7291i$	$-0.6000 \pm 4.0000i$
$-0.0662 \pm 4.0679i$	$-0.7000 \pm 5.0000i$
$-0.2114 \pm 7.2673i$	$-0.8000 \pm 8.0000i$
$-0.3772 \pm 9.7038i$	$-0.9000 \pm 10.000i$

mum gain design results in a smaller gain matrix norm, but the resulting closed-loop matrix may have robustness problems. The second design, on the other hand, yields a substantially more robust closed-loop system with moderately higher gains. Although, none of the design are globally optimum (optimizations are performed at each eigenvalue assignment only), they can provide reasonably efficient and robust controllers.

Output Feedback

The space truss of Fig. 1 is also used to demonstrate the eigenvalue assignment technique using output feedback. There are four displacement sensors along the degrees of freedom 6, 7, 16, 17, and four velocity sensors along the degrees of freedom 10, 11, 20, and 21 to provide the output measurements. These eight sensors and the six actuators allow the freedom to assign six pairs of eigenvalues. Three pairs of the closed-loop eigenvalues are arbitrarily chosen as $-0.7 \pm 10i$, $-0.8 \pm 12i$, and $-1.0 \pm 15i$, and the remaining three pairs are selected as the first three open-loop pairs $-0.0064 \pm 1.2617i$, $-0.0556 \pm 3.7291i$, and -0.0662 ± 4.0679.

Following the developments given in Eqs. (40–56), the gain matrix G is subsequently computed from Eqs. (57) and (58), having a norm of 141.93. The resulting closed-loop system is stable (all the closed-loop eigenvalues lie on the left of the imaginary axis). It is noted, however, that the proposed output feedback eigenvalue assignment technique can only enhance the possibility of achieving a stable solution by imposing constraints on some of the most sensitive open-loop eigenvalues and by no means guarantees stable closed-loop solutions. The coefficients c_k of Eq. (50) were arbitrarily chosen to generate a set of $2s$ linearly independent eigenvectors, which accounts for a moderately large magnitude of the norm of the gain matrix G. Possibly, optimal projection techniques discussed in Ref. 13 or iterative techniques outlined in Ref. 8 may be used to identify a set of coefficients that yield better solutions in terms of lower gain norms or good closed-loop conditioning.

Concluding Remarks

A novel and efficient approach for the eigenvalue assignment via either full-state or output feedback for large first-order, time-invariant systems has been developed. Since the approach assigns one eigenvalue at a time without shifting the remaining eigenvalues, it is especially efficient when the number of assigned eigenvalues is smaller than the order of the system. Moreover, the formulations are derived using the additional freedom of multi-inputs to minimize a specific norm of the gain matrix and/or a robustness index for the closed-

loop system. The minimum gain is attractive particularly for space applications due to limited available power. On the other hand, the robustness of the closed-loop system requires special attention particularly for large-order complex space structures where structural uncertainties may be significant. Both issues were addressed in the paper including suggestions on how to approach the problem.

Numerical examples have been presented for a flexible space structure for which desired closed-loop eigenvalues are assigned using both full-state feedback and output feedback algorithms. Both algorithms were also successfully applied to other large space structures and compared with other existing methods. Results to date indicate that both algorithms are computationally sound and stable and can be used as viable control design tools.

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